

Chromatic bounds for some classes of $2K_2$ -free graphs

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Abstract

A hereditary class \mathcal{G} of graphs is χ -bounded if there is a χ -binding function, say f such that $\chi(G) \leq f(\omega(G))$, for every $G \in \mathcal{G}$, where $\chi(G)$ ($\omega(G)$) denote the chromatic (clique) number of G . It is known that for every $2K_2$ -free graph G , $\chi(G) \leq \binom{\omega(G)+1}{2}$, and the class of $(2K_2, 3K_1)$ -free graphs does not admit a linear χ -binding function. In this paper, we are interested in classes of $2K_2$ -free graphs that admit a linear χ -binding function. We show that the class of $(2K_2, H)$ -free graphs, where $H \in \{K_1 + P_4, K_1 + C_4, \overline{P_2 \cup P_3}, HVN, K_5 - e, K_5\}$ admits a linear χ -binding function. Also, we show that some superclasses of $2K_2$ -free graphs are χ -bounded.

Keywords. Chromatic number; clique number; graph classes; $2K_2$ -free graphs.

1 Introduction

All graphs in this paper are simple, finite and undirected. For notation and terminology that are not defined here, we refer to West [20]. Let P_n , C_n , K_n denote the induced path, induced cycle and complete graph on n vertices respectively. Let $K_{p,q}$ be the complete bipartite graph with classes of size p and q . If \mathcal{F} is a family of graphs, a graph G is said to be \mathcal{F} -free if it contains no induced subgraph isomorphic to any member of \mathcal{F} . If G_1 and G_2 are two vertex disjoint graphs, then their *union* $G_1 \cup G_2$ is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Similarly, their *join* $G_1 + G_2$ is the graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(x, y) \mid x \in V(G_1), y \in V(G_2)\}$. For any positive integer k , kG denotes the union of k graphs each isomorphic to G . For a graph G , the complement of G is denoted by \overline{G} .

A *proper coloring* (or simply *coloring*) of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. The minimum number of colors required to color G is called the *chromatic number* of G , and is denoted by $\chi(G)$. A *clique* in a graph G is a set of vertices that are pairwise adjacent in G . The *clique number* of G , denoted by $\omega(G)$, is the size of a maximum clique in G . Obviously, for any graph G , we have $\chi(G) \geq \omega(G)$. The existence of triangle-free graphs with large chromatic number (see [16] for a construction of

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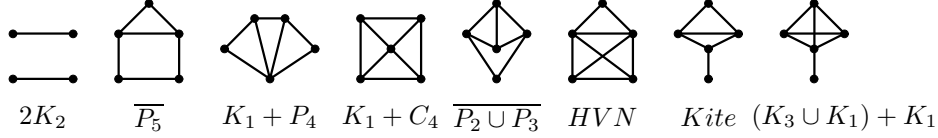


Figure 1: Some special graphs.

such graphs) shows that for a general class of graphs, there is no upper bound on the chromatic number as a function of clique number.

A graph G is called *perfect* if $\chi(H) = \omega(H)$, for every induced subgraph H of G ; otherwise it is called *imperfect*. A hereditary class \mathcal{G} of graphs is said to be χ -*bounded* [10] if there exists a function f (called a χ -*binding function* of \mathcal{G}) such that $\chi(G) \leq f(\omega(G))$, for every $G \in \mathcal{G}$. If \mathcal{G} is the class of H -free graphs for some graph H , then f is denoted by f_H . We refer to [17] for an extensive survey of χ -bounds for various classes of graphs.

The class of $2K_2$ -free graphs and its related classes have been well studied in various contexts in the literature; see [2]. Here, we would like to focus on showing χ -binding functions for some classes of graphs related to $2K_2$ -free graphs. Wagon [19] showed that the class of mK_2 -free graphs admits an $O(x^{2m-2})$ χ -binding function for all $m \geq 1$. In particular, he showed that $f_{2K_2}(x) = \binom{x+1}{2}$, and the best known lower bound is $\frac{R(C_4, K_{x+1})}{3}$, where $R(C_4, K_{x+1})$ denotes the smallest k such that every graph on k vertices contains either a clique of size $x+1$ or the complement of the graph contains a C_4 [10]. This lower bound is non-linear because Chung [7] showed that $R(C_4, K_t)$ is at least $t^{1+\epsilon}$ for some $\epsilon > 0$. It is interesting to note that Brause et al. [3] showed that the class of $(2K_2, 3K_1)$ -free graphs does not admit a linear χ -binding function. It follows that the class of $(2K_2, H)$ -free graphs, where H is any $2K_2$ -free graph with independence number $\alpha(H) \geq 3$, does not admit a linear χ -binding function.

Here we are interested in classes of $2K_2$ -free graphs that admit a linear χ -binding function, in particular, some classes of $2K_2$ -free graphs that admit a ‘special’ linear χ -binding function $f(x) = x + c$, where c is an integer, that is, $2K_2$ -free graphs G such that $\chi(G) \leq \omega(G) + c$. If $c = 1$, then this special upper bound is called the *Vizing bound* for the chromatic number, and is well studied in the literature; see [13, 17] and the references therein. Brause et al. [3] showed that if G is a connected $(2K_2, K_{1,3})$ -free graph with independence number $\alpha(G) \geq 3$, then G is perfect. It follows from a result of [12] that if G is a $(2K_2, \text{paw})$ -free graph, then either G is perfect or $\chi(G) = 3$ and $\omega(G) = 2$ (see also [3]). Nagy and Szentmiklóssy (see [10]) showed that if G is a $(2K_2, K_4)$ -free graph, then $\chi(G) \leq 4$. Blaszik et al. [1] and independently Gyárfás [10] showed that if G is $(2K_2, C_4)$ -free graph, then $\chi(G) \leq \omega(G) + 1$, and the equality holds if and only if G is a split-graph. It follows from a result of [13] that if G is a $(2K_2, K_4 - e)$ -free graph, then $\chi(G) \leq \omega(G) + 1$. Fouquet et al. [9] showed that if G is a $(2K_2, \overline{P_5})$ -free graph, then $\chi(G) \leq \left\lfloor \frac{3\omega(G)}{2} \right\rfloor$, and the bound is tight. Brause et al. [3] showed that if G is a $(2K_2, K_1 + P_4)$ -free graph, then $\chi(G) \leq 2\omega(G)$.

In this paper, by using structural results, we show that the class of $(2K_2, H)$ -free graphs, where $H \in \{K_1 + P_4, K_1 + C_4, \overline{P_2 \cup P_3}, HVN, K_5 - e\}$ admits a special linear χ -binding function $f(x) = x + c$, where c is an integer; see Figure 1. We also show that the class of $(2K_2, K_5)$ -free

Graph class \mathcal{C}	χ -bound for $G \in \mathcal{C}$	
$(2K_2, \overline{P_5})$ -free graphs	$\lfloor \frac{3\omega(G)}{2} \rfloor$	[9]
$(2K_2, C_5)$ -free graphs	$\omega(G)^{3/2}$	[11]
$(2K_2, K_1 + P_4)$ -free graphs	$\omega(G) + 2$	(Corollary 1)
$(2K_2, K_1 + C_4)$ -free graphs	$\omega(G) + 5$	(Corollary 2)
$(2K_2, \overline{P_2 \cup P_3})$ -free graphs	$\omega(G) + 1$	(Corollary 3)
$(2K_2, HVN)$ -free graphs	$\omega(G) + 3$	(Corollary 4)
$(2K_2, K_5 - e)$ -free graphs	$\omega(G) + 4$	(Corollary 5)
$(2K_2, K_5)$ -free graphs	$2\omega(G) + 1 \leq 9$	(Corollary 6)
$(2K_2, X)$ -free graphs	$\binom{\omega(G)+1}{2}$	[19]

Table 1: Known chromatic bounds for $(2K_2, H)$ -free graphs, where H is any $2K_2$ -free graph on 5 vertices with $\alpha(H) = 2$, and the graph $X \in \{Kite, K_4 \cup K_1, (K_3 \cup K_1) + K_1\}$.

graphs admits a linear χ -binding function. Table 1 shows the known chromatic bounds for a $(2K_2, H)$ -free graph G , where H is any $2K_2$ -free graph on 5 vertices with $\alpha(H) = 2$. Some of the cited bounds are consequences of much stronger results available in the literature. Finally, we show χ -binding functions for some superclasses of $2K_2$ -free graphs.

2 Notation, terminology, and preliminaries

Let G be a graph, with vertex-set $V(G)$ and edge-set $E(G)$. For $x \in V(G)$, $N(x)$ denotes the set of all neighbors of x in G . For any two disjoint subsets $S, T \subseteq V(G)$, $[S, T]$ denotes the set of edges $\{e \in E(G) \mid e \text{ has one end in } S \text{ and the other in } T\}$. Also, for $S \subseteq V(G)$, let $G[S]$ denotes the subgraph induced by S in G , and for convenience we simply write $[S]$ instead of $G[S]$. Note that if H_1 and H_2 are any two graphs, and if G is (H_1, H_2) -free, then \overline{G} is $(\overline{H_1}, \overline{H_2})$ -free. For any integer k , we write $[k]$ to denote the set $\{1, 2, \dots, k\}$.

A *diamond* or a $K_4 - e$ is the graph with vertex set $\{a, b, c, d\}$ and edge set $\{ab, bc, cd, ad, bd\}$. A *paw* is the graph with vertex set $\{a, b, c, d\}$ and edge set $\{ab, bc, ac, ad\}$. See Figure 1 for some of the other special graphs used in this paper.

A graph G is a *split graph* if its vertex set $V(G)$ can be partitioned into two sets V_1 and V_2 such that V_1 is a clique and V_2 is an independent set. In [8], Földes and Hammer showed that a graph G is a split graph if and only if G is $(2K_2, C_4, C_5)$ -free. A graph G is a *pseudo-split graph* [15] if G is $(2K_2, C_4)$ -free. The class of pseudo-split graphs generalizes the class of split graphs.

A *k-clique covering* of a graph G is a partition (V_1, V_2, \dots, V_k) of $V(G)$ such that V_i is a clique, for each $i \in \{1, 2, \dots, k\}$. The *clique covering number* of the graph G , denoted by $\theta(G)$, is the minimum integer k such that G admits a k -clique covering. An *independent/stable* set in a graph G is a set of vertices that are pairwise non-adjacent in G . The *independence number* of G , denoted by $\alpha(G)$, is the size of a maximum independent set in G . Clearly, for any graph G , we have $\chi(G) = \theta(\overline{G})$ and $\omega(G) = \alpha(\overline{G})$.

We also use the following known results:

(R1) **Seinsche ([18]):** *If G_1 and G_2 are P_4 -free, then $G_1 \cup G_2$ and $G_1 + G_2$ are P_4 -free.*

- (R2) **Seinsche ([18])**: Every P_4 -free graph is perfect.
- (R3) **Chudnovsky et al. ([6])** (THE STRONG PERFECT GRAPH THEOREM (**SPGT**)): A graph is perfect if and only if it contains no odd hole (chordless cycle) of length at least 5 and no odd antihole (complement graph of a hole) of length at least 5.
- (R4) **Choudum et al. ([5])**: Let \mathcal{G} and \mathcal{F} be hereditary classes of graphs where \mathcal{F} admits a linear χ -binding function. If there exists a constant k such that for any $G \in \mathcal{G}$, $V(G)$ can be partitioned into k subsets V_1, V_2, \dots, V_k , where $[V_i] \in \mathcal{F}$ for each $i \in \{1, \dots, k\}$, then \mathcal{G} has a linear χ -binding function.
- (R5) **Blazsik et al. ([1])**: For every pseudo-split graph G , $\chi(G) \leq \omega(G) + 1$.
- (R6) **Brause et al. ([3])**: For every $(2K_2, \text{paw})$ -free graph G , $\chi(G) \leq \omega(G) + 1$.
- (R7) **Karthick and Maffray ([13])**: For every $(2K_2, \text{diamond})$ -free graph G , $\chi(G) \leq \omega(G) + 1$.

3 Linearly χ -bounded $2K_2$ -free graphs

In this section, we show that the class of $(2K_2, H)$ -free graphs, where $H \in \{K_1 + P_4, K_1 + C_4, \overline{P_2 \cup P_3}, K_5 - e, HVN, K_5\}$ is linearly χ -bounded. Note that the class of $(2K_2, K_1 + C_4)$ -free graphs and the class of $(2K_2, \overline{P_2 \cup P_3})$ -free graphs generalize the class of $(2K_2, C_4)$ -free graphs or pseudo-split graphs. Also the class of $(2K_2, K_5 - e)$ -free graphs, the class of $(2K_2, \overline{P_2 \cup P_3})$ -free graphs, and the class of $(2K_2, HVN)$ -free graphs generalize the class of $(2K_2, K_4 - e)$ -free graphs.

3.1 The class of $(2K_2, K_1 + P_4)$ -free graphs

First we prove a structure theorem for the complement graph of a $(2K_2, K_1 + P_4)$ -free graph.

Theorem 1 *Let G be an imperfect $(P_4 \cup K_1, C_4)$ -free graph. Then G is connected and there exists a partition (V_1, V_2, V_3) of $V(G)$ such that*

- (i) V_1 induces a P_4 -free subgraph of G , and
- (ii) V_2 and V_3 are cliques.

Proof. Let G be an imperfect $(P_4 \cup K_1, C_4)$ -free graph. Since G is $(P_4 \cup K_1)$ -free, G contains no hole of length at least 7, and since G is C_4 -free, G contains no anti-hole of length at least 7. Thus, it follows from SPGT [6] that G contains a 5-hole (hole of length 5), say C with vertex-set $\{v_1, v_2, v_3, v_4, v_5\}$, and edge-set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Throughout this proof, we take all the subscripts of v_i to be modulo 5.

Claim 1 *Any vertex $x \in V(G) \setminus V(C)$ is adjacent to at least two vertices in C .*

Proof of Claim 1. Suppose not. If x is not adjacent to any of the vertices in C , or if x is adjacent to exactly one vertex in C , say to v_1 , then $\{v_2, v_3, v_4, v_5, x\}$ induces a $P_4 \cup K_1$ in G , which is a contradiction. \diamond

By Claim 1, G is connected.

Claim 2 *If $x \in V(G) \setminus V(C)$, then $[N(x) \cap V(C)]$ is isomorphic to a member of $\{K_2, P_3, C_5\}$.*

Proof of Claim 2. Suppose not. Then by Claim 1, $N(x) \cap V(C)$ is either $\{v_i, v_{i+3}\}$ or $\{v_i, v_{i+1}, v_{i+3}\}$ or $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$, for some i . But then in all the cases, $\{v_i, x, v_{i+3}, v_{i+4}\}$ induces a C_4 in G , which is a contradiction. \diamond

For $i \in [5]$, let:

$$\begin{aligned} A_i &= \{x \in V(G) \setminus V(C) \mid N(x) \cap V(C) = \{v_i, v_{i+1}\}\}, \\ B_i &= \{x \in V(G) \setminus V(C) \mid N(x) \cap V(C) = \{v_{i-1}, v_i, v_{i+1}\}\}, \\ D &= \{x \in V(G) \setminus V(C) \mid N(x) \cap V(C) = V(C)\}. \end{aligned}$$

Moreover, let $A = A_1 \cup \dots \cup A_5$, and $B = B_1 \cup \dots \cup B_5$. Then by Claims 1 and 2, we have $V(G) = V(C) \cup A \cup B \cup D$.

Claim 3 *For each $i \in [5]$ ($i \bmod 5$), the following hold:*

- (i) A_i induces a P_4 -free subgraph of G .
- (ii) $[A_i, A_{i+1}]$ is complete.
- (iii) If A_i is not a clique, then A_{i+1} and A_{i-1} are cliques in G .
- (iv) If $A_i \neq \emptyset$, then $A_{i+2} = \emptyset = A_{i-2}$.

Proof of Claim 3. (i) Suppose to the contrary that $[A_i]$ contains an induced P_4 , say P . Then by the definition of A_i , $V(P) \cup \{v_{i+3}\}$ induces a $P_4 \cup K_1$ in G , a contradiction. So (i) holds.

Suppose that (ii) does not hold. Then there exist vertices $x \in A_i$ and $y \in A_{i+1}$ such that $xy \notin E(G)$. But, then $\{v_{i+3}, v_{i+4}, v_i, x, y\}$ induces a $P_4 \cup K_1$ in G , a contradiction. So (ii) holds.

Suppose that (iii) does not hold. Then there exist two non-adjacent vertices a and b in A_i , and up to symmetry, two non-adjacent vertices x and y in A_{i+1} . But then since $[A_i, A_{i+1}]$ is complete (by (ii)), $\{a, b, x, y\}$ induces a C_4 in G , a contradiction. So (iii) holds.

Suppose that (iv) does not hold. Then there exist vertices $x \in A_i$ and $y \in A_{i+2} \cup A_{i-2}$. By symmetry, we may assume that $y \in A_{i+2}$. But then $\{x, v_{i+1}, v_{i+2}, y\}$ induces a C_4 in G , if $xy \in E(G)$, and $\{x, v_{i+1}, v_{i+2}, y, v_{i+4}\}$ induces a $P_4 \cup K_1$ in G , if $xy \notin E(G)$, a contradiction. So (iv) holds. \diamond

Claim 4 *For each $i \in [5]$ ($i \bmod 5$), the following hold:*

- (i) $\{v_i\} \cup B_i \cup D$ is a clique.
- (ii) $B_i \cup B_{i+1}$ induces a P_4 -free subgraph of G .
- (iii) $[B_i, B_{i+2}] = \emptyset = [B_i, B_{i-2}]$.

Proof of Claim 4. Suppose that (i) does not hold. Then there exist vertices $x, y \in B_i \cup D$ such that $xy \notin E(G)$. But, then $\{x, v_{i-1}, y, v_{i+1}\}$ induces a C_4 in G , a contradiction. So (i) holds.

Suppose that (ii) does not hold. Suppose that $[B_i \cup B_{i+1}]$ contains an induced P_4 , say P . Then by the definition of B_i 's, $V(P) \cup \{v_{i+3}\}$ induces a $P_4 \cup K_1$ in G , a contradiction. So (ii) holds.

Suppose that (iii) does not hold. Then there exist vertices $x \in B_i$ and $y \in B_{i+2} \cup B_{i-2}$ such that $xy \in E(G)$. By symmetry, we may assume that $y \in B_{i+2}$. But, then $\{x, v_{i+4}, v_{i+3}, y\}$ induces a C_4 in G , a contradiction. So (ii) holds. \diamond

Claim 5 *For each $i \in [5]$ ($i \bmod 5$), the following hold:*

- (i) $[A_i, B_i \cup B_{i+1}]$ are complete.
- (ii) $[A_i, B_{i+3}] = \emptyset$.

Proof of Claim 5. Suppose that (i) does not hold. Then there exist vertices $x \in A_i$ and $y \in B_i \cup B_{i+1}$ such that $xy \notin E(G)$. But, then $\{v_{i+2}, v_{i+3}, v_{i+4}, y, x\}$ induces a $P_4 \cup K_1$ in G , a contradiction. So (i) holds.

Suppose that (ii) does not hold. Then there exist vertices $x \in A_i$ and $y \in B_{i+3}$ such that $xy \in E(G)$. But, then $\{x, v_i, v_{i+4}, y\}$ induces a C_4 in G , a contradiction. So (ii) holds. \diamond

By Claim 3(iv), we may assume that $A \setminus (A_1 \cup A_2) = \emptyset$. Now, by using the above claims, we prove the theorem in two cases (the other cases are symmetric).

Case 1. *Suppose that A_1 is not a clique.*

Then by Claim 3(iii), A_2 is a clique. Then we define $V_1 := \{v_1, v_2, v_4\} \cup A_1 \cup B_1 \cup B_2 \cup B_4$, $V_2 := \{v_3\} \cup A_2 \cup B_3$, and $V_3 := \{v_5\} \cup B_5 \cup D$, and we have the following:

Claim 6 V_1 induces a P_4 -free subgraph of G .

Proof of Claim 6. By Claim 3(i), $[A_1]$ is P_4 -free, and by Claim 4(ii), $[B_1 \cup B_2]$ is P_4 -free. Also, $[A_1, B_1 \cup B_2]$ is complete (by Claim 5(i)). So, by the definition of A_1 , B_1 and B_2 , by (R1), $[\{v_1, v_2\} \cup A_1 \cup B_1 \cup B_2]$ is P_4 -free, since it is a join of two P_4 -free graphs, namely $[A_1]$ and $[\{v_1, v_2\} \cup B_1 \cup B_2]$. Now, $B_4 \cup \{v_4\}$ is a clique (by Claim 4(i)), $[B_4, A_1] = \emptyset$ (by Claim 5(ii)), and $[B_4, B_1] = \emptyset = [B_4, B_2]$ (by Claim 4(iii)). Thus, by (R1), $[V_1]$ is P_4 -free, as it is a union of two disjoint P_4 -free graphs, namely $[\{v_1, v_2\} \cup A_1 \cup B_1 \cup B_2]$ and $[B_4 \cup \{v_4\}]$. Hence the claim holds. \diamond

Claim 7 V_2 and V_3 are cliques in G .

Proof of Claim 7. The proof follows by the definition of A_2 , B_3 and B_5 , and by Claims 4(i) and 5(i). \diamond

So, by Claims 6 and 7, we see that (V_1, V_2, V_3) is a required partition of $V(G)$.

Case 2. *Suppose that both A_1 and A_2 are cliques.*

The proof is similar to that of Case 1.

Hence the theorem is proved. \square

The following corollary is an improvement over that in [3], where it is shown that for every $(2K_2, K_1 + P_4)$ -free graph G , $\chi(G) \leq 2\omega(G)$.

Corollary 1 *Let G be a $(2K_2, K_1 + P_4)$ -free graph. Then $\chi(G) \leq \omega(G) + 2$.*

Proof. Consider the complement H of G . Then H is a $(P_4 \cup K_1, C_4)$ -free graph.

If H is perfect, then $\theta(H) = \alpha(H)$, and the corollary holds.

If H is imperfect, then by Theorem 1, H is connected and there exists a partition (V_1, V_2, V_3) of H such that V_1 induces a P_4 -free subgraph of H , and V_2 and V_3 are cliques in H . So, $\theta(H) \leq \theta([V_1]) + \theta([V_2 \cup V_3])$. Since every P_4 -free graph is perfect (by (R2)), we have $\theta(H) \leq \alpha([V_1]) + 2 \leq \alpha(H) + 2$, and the corollary follows. \square

3.2 The class of $(2K_2, K_1 + C_4)$ -free graphs

First we prove a structure theorem for the class of $(2K_2, K_1 + C_4)$ -free graphs.

Theorem 2 *Let G be a connected $(2K_2, K_1 + C_4)$ -free graph. Then G is either a pseudo-split graph or there exists a partition (V_1, \dots, V_6) of $V(G)$ such that*

- (i) $[V_1]$ is either a pseudo-split graph of G with $\omega([V_1]) \leq \omega(G) - 1$ or the complement of a bipartite graph of G , and
- (ii) V_i is an independent set, for each $i \in \{2, \dots, 6\}$. Moreover, if V_1 induces a pseudo-split graph of G , then $V_5 = \emptyset = V_6$.

Proof. Let G be a connected $(2K_2, K_1 + C_4)$ -free graph.

If G is C_4 -free, then G is a pseudo-split graph, and the theorem holds.

Suppose that G contains an induced C_4 , say C with vertex-set $L_0 := \{v_1, v_2, v_3, v_4\}$, and edge-set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Define sets $L_1 := \{y \in V(G) \setminus L_0 \mid y \text{ has a neighbor in } L_0\}$ and $L_2 := V(G) \setminus (L_0 \cup L_1)$. Throughout this proof, we take all the subscripts of v_i to be modulo 4.

Claim 1 *If $x \in L_1$, then $|N(x) \cap L_0| \in \{1, 2, 3\}$.*

Proof of Claim 1. Otherwise, $L_0 \cup \{x\}$ induces a $K_1 + C_4$ in G , a contradiction. \diamond

So, for any $x \in L_1$, there exists an index $j \in [4]$ such that $xv_j \in E(G)$ and $xv_{j+1} \notin E(G)$. For $i \in [4]$, let:

$$\begin{aligned} W_i &= \{x \in L_1 \mid N(x) \cap L_0 = \{v_i\}\}, \\ X_i &= \{x \in L_1 \mid N(x) \cap L_0 = \{v_i, v_{i+1}\}\}, \\ Y_1 &= \{x \in L_1 \mid N(x) \cap L_0 = \{v_1, v_3\}\}, \\ Y_2 &= \{x \in L_1 \mid N(x) \cap L_0 = \{v_2, v_4\}\}, \\ Z_i &= \{x \in L_1 \mid N(x) \cap L_0 = \{v_{i-1}, v_i, v_{i+1}\}\}. \end{aligned}$$

Moreover, let $W = W_1 \cup \dots \cup W_4$, $X = X_1 \cup \dots \cup X_4$, and $Z = Z_1 \cup \dots \cup Z_4$. Then, by Claim 1, $V(G) = L_0 \cup W \cup X \cup Y_1 \cup Y_2 \cup Z \cup L_2$. Now:

Claim 2 *The following hold:*

- (i) *If $x \in W \cup X$, then $N(x) \cap L_2 = \emptyset$.*

(ii) L_2 is an independent set.

Proof of Claim 2. (i) We may assume that $x \in W_1 \cup X_1$, and suppose to the contrary that $y \in N(x) \cap L_2$. Then $\{y, x, v_3, v_4\}$ induces a $2K_2$ in G , a contradiction. So (i) holds.

Suppose that (ii) does not hold. Then there exist two adjacent vertices, say x and y in L_2 . But, then $\{x, y, v_1, v_2\}$ induces a $2K_2$ in G , a contradiction. So (ii) holds. \diamond

Claim 3 For each $i \in [4]$ ($i \bmod 4$), the following hold:

- (i) $W_i \cup W_{i+1} \cup X_i$ is an independent set.
- (ii) If $X_i \neq \emptyset$, then either $X_{i+1} = \emptyset$ or $X_{i+2} = \emptyset$.
- (iii) If $Z_i \neq \emptyset$, then $Z_{i+2} = \emptyset$.

Proof of Claim 3. We prove the claim for $i = 1$.

(i) Suppose to the contrary that there exist two adjacent vertices, say x and y in $W_1 \cup W_2 \cup X_1$. Then $\{x, y, v_3, v_4\}$ induces a $2K_2$ in G , a contradiction. So (i) holds.

Suppose that (ii) does not hold. Then there exist vertices $x_1 \in X_1$, $x_2 \in X_2$ and $x_3 \in X_3$. Then since $\{x_1, v_1, x_2, v_3\}$ or $\{x_1, v_1, x_3, v_3\}$ or $\{x_2, v_2, x_3, v_4\}$ do not induce a $2K_2$ in G , $\{x_1, x_2, x_3\}$ induces a triangle in G . But, then $\{x_1, v_2, v_3, x_3, x_2\}$ induces a $K_1 + C_4$ in G , a contradiction. So (ii) holds.

Suppose that (iii) does not hold. Then there exist vertices $x \in Z_1$ and $y \in Z_3$. But then $\{v_1, v_2, y, v_4, x\}$ induces a $K_1 + C_4$ in G , if $xy \in E(G)$, or $\{v_1, x, y, v_3\}$ induces a $2K_2$ in G , if $xy \notin E(G)$, a contradiction. So (iii) holds. \diamond

Claim 4 For each $i \in \{1, 2\}$, Y_i is a union of a clique and an independent set.

Proof of Claim 4. We prove the claim for $i = 1$. First, we show that $[Y_1]$ is P_3 -free. Suppose to the contrary that $[Y_1]$ contains an induced P_3 , say P . Then by the definition of Y_1 , $V(P) \cup \{v_1, v_3\}$ induces a $K_1 + C_4$ in G , a contradiction. So, $[Y_1]$ is P_3 -free, and hence it is a union of cliques. Then since G is $2K_2$ -free, it follows that Y_1 is a union of a clique and an independent set, and the claim holds. \diamond

By Claim 4, for each $i \in \{1, 2\}$, we define $Y_i := Y'_i \cup Y''_i$, where Y'_i is a clique, and Y''_i is an independent set.

Claim 5 For each $i \in \{1, 2\}$, $[Z_i \cup Z_{i+2}, Y_{3-i}] = \emptyset$.

Proof of Claim 5. We prove the claim for $i = 1$. Suppose to the contrary that there exist vertices, say $z \in Z_1 \cup Z_3$ and $y \in Y_2$ such that $zy \in E(G)$. But, then $\{v_1, v_2, y, v_4, z\}$ or $\{v_2, v_3, v_4, y, z\}$ induces a $K_1 + C_4$ in G , a contradiction. So the claim holds. \diamond

Claim 6 For each $i \in \{1, 2\}$, if $Z_i \cup Z_{i+2} \neq \emptyset$, then Y_{3-i} is an independent set.

Proof of Claim 6. We prove the claim for $i = 1$. Let $z \in Z_1 \cup Z_3$. Up to symmetry, we may assume that $z \in Z_1$. By Claim 5, $[\{z\}, Y_2] = \emptyset$. Now, we show that Y_2 is an independent set. Suppose to the contrary that there exist adjacent vertices, say p and q in Y_2 . Then since $[\{z\}, Y_2] = \emptyset$, we have $zp \notin E(G)$ and $zq \notin E(G)$. But, then $\{z, v_1, p, q\}$ induces a $2K_2$ in G , a contradiction. So the claim holds. \diamond

Now, by using Claim 3(iii), we prove the theorem in two cases.

Case 1. Suppose that $Z_i = \emptyset$, for every $i \in [4]$.

By Claim 3(ii) and by symmetry, we may assume that either $X_2 \cup X_4 = \emptyset$ or $X_3 \cup X_4 = \emptyset$. Then we define $V_1 := Y'_1 \cup Y'_2 \cup \{v_1, v_2\}$, $V_2 := Y''_1 \cup \{v_4\}$, $V_3 := Y''_2 \cup \{v_3\}$, $V_4 := W_1 \cup W_2 \cup X_1 \cup L_2$. Further: If $X_2 \cup X_4 = \emptyset$, then we define $V_5 := W_3 \cup W_4 \cup X_3$ and $V_6 := \emptyset$; and if $X_3 \cup X_4 = \emptyset$, then we define $V_5 := W_3 \cup X_2$ and $V_6 := W_4$.

Now, by Claims 2 and 3(i), and by the definition of Y'_i 's and Y''_i 's, we see that $[V_1]$ is isomorphic to the complement of a bipartite graph, and V_i 's are independent sets, for each $i \in \{2, \dots, 6\}$. So, (V_1, \dots, V_6) is a required partition of $V(G)$.

Case 2. Suppose that $Z_i \cup Z_{i+1} \neq \emptyset$, for exactly one $i \in [4]$.

We may assume up to symmetry that $i = 1$ and $Z_1 \neq \emptyset$. Then by Claim 6, Y_2 is an independent set. Then, we define $V_1 := N(v_1)$, $V_2 := W_2 \cup X_2 \cup L_2$, $V_3 := W_3 \cup W_4 \cup X_3$, $V_4 := Y_2 \cup \{v_1, v_3\}$, $V_5 := \emptyset$ and $V_6 := \emptyset$. Then since G is $(K_1 + C_4)$ -free, V_1 induces a pseudo-split graph in G . Also, $\omega([V_1]) \leq \omega(G) - 1$. So, by Claims 2 and 3(i), we see that V_i 's are independent sets, for each $i \in \{2, 3, 4\}$, and hence (V_1, \dots, V_6) is a required partition of $V(G)$.

This completes the proof of the theorem. \square

Corollary 2 Let G be a $(2K_2, K_1 + C_4)$ -free graph. Then $\chi(G) \leq \omega(G) + 5$.

Proof. Let G be a $(2K_2, K_1 + C_4)$ -free graph. We may assume that G is connected. We use Theorem 2. If G is a pseudo-split graph, then, by (R5), $\chi(G) \leq \omega(G) + 1$. So, suppose that $V(G)$ admits a partition as in Theorem 2. Now:

(a) Suppose that V_1 induces a pseudo-split graph with $\omega([V_1]) \leq \omega(G) - 1$. Then $V_5 = \emptyset = V_6$. So, $\chi(G) \leq \chi([V_1]) + 3$. Then by (R4) and (R5), $\chi(G) \leq \omega([V_1]) + 1 + 3 \leq \omega(G) + 3$.

(b) Suppose that $[V_1]$ is isomorphic to the complement of a bipartite graph, and V_i 's are independent sets, for each $i \in \{2, \dots, 6\}$. Then since $[V_1]$ is perfect, it follows by (R4) that $\chi(G) \leq \omega([V_1]) + 5 \leq \omega(G) + 5$. Hence the corollary is proved. \square

3.3 The class of $(2K_2, \overline{P_2 \cup P_3})$ -free graphs

We use the structure theorem for $(P_2 \cup P_3, C_4)$ -free graphs proved in [4]. We need the following definition.

Let G be a graph on n vertices v_1, v_2, \dots, v_n , and let H_1, H_2, \dots, H_n be any n vertex disjoint graphs. Then an *expansion* $G(H_1, H_2, \dots, H_n)$ of G [4] is the graph obtained from G by

- (i) replacing the vertex v_i of G by H_i , $i = 1, 2, \dots, n$, and
- (ii) joining the vertices $x \in H_i$, $y \in H_j$ iff v_i and v_j are adjacent in G .

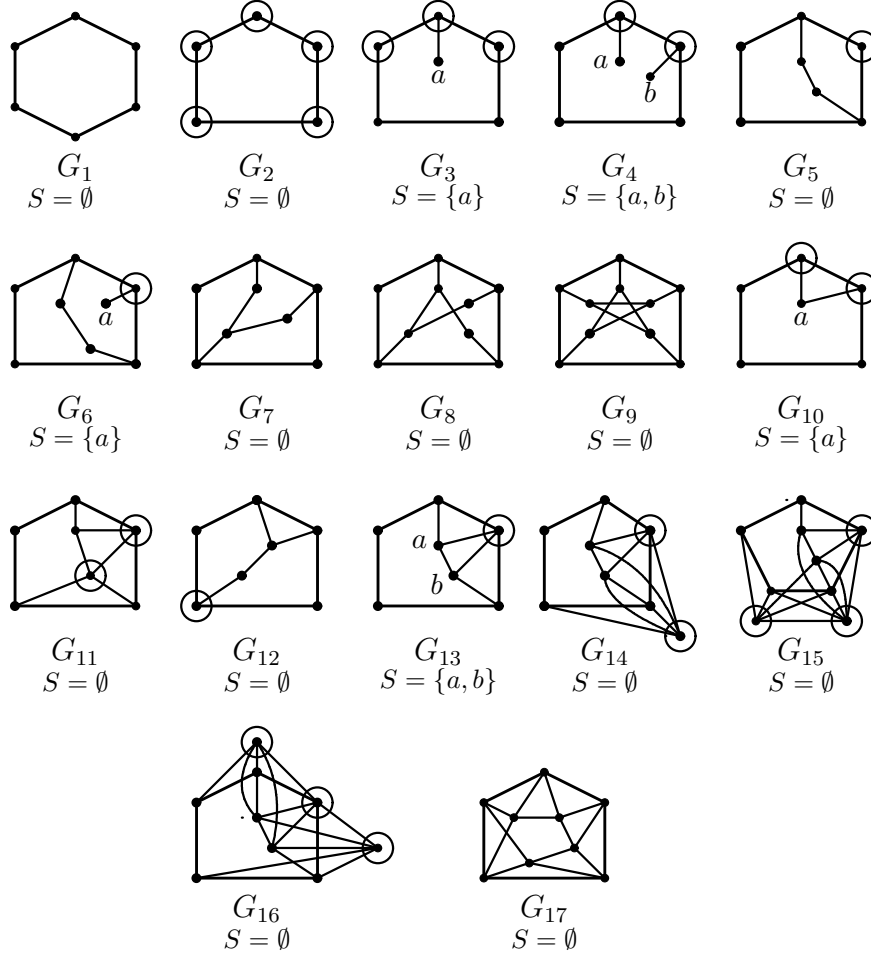


Figure 2: Basic graphs used in Theorem 3.

An expansion is also called a *composition*; see [20]. If H_i 's are complete, it is called a *complete expansion* of G . By a result of Lovász [14], if G, H_1, H_2, \dots, H_n are perfect, then $G(H_1, H_2, \dots, H_n)$ is perfect.

Theorem 3 ([4]) *If G is a connected $(P_2 \cup P_3, C_4)$ -free graph, then G is either chordal or there exists a partition (V_1, V_2, V_3) of $V(G)$ such that (1) $[V_1] \cong K_m^c$, for some $m \geq 0$, (2) $[V_2] \cong K_t$, for some $t \geq 0$, (3) $[V_3]$ is isomorphic to a graph obtained from one of the basic graphs G_t ($1 \leq t \leq 17$) shown in Figure 2 by expanding each vertex indicated in circle by a complete graph (of order ≥ 1), (4) $[V_1, V_3] = \emptyset$, and (5) $[V_2, V_3 \setminus S]$ is complete (see Figure 2 for the set S).*

For $t \in [17]$, let \mathcal{G}_t denote the class of graphs obtained from G_t (see Figure 2) by the operations stated in Theorem 3.

Corollary 3 *Let G be a $(2K_2, \overline{P_2 \cup P_3})$ -free graph. Then $\chi(G) \leq \omega(G) + 1$.*

Proof. Consider the complement H of G . Then H is a $(P_2 \cup P_3, C_4)$ -free graph.

If H is chordal, then H is perfect and so $\theta(H) = \alpha(H)$, and the corollary holds.

Suppose that H is not chordal. Let H_1, H_2, \dots, H_k ($k \geq 1$) denote the components of H . Then since H is not chordal and since H is $(P_2 \cup P_3, C_4)$ -free, by Theorem 3, we may assume that there exists a component, say H_1 of G such that $V(H_1)$ admits a partition (V_1, V_2, V_3) as in Theorem 3 where $[V_3]$ contains either a C_5 or a C_6 , and $[V_3] \in \mathcal{G}_t$, for $t \in [17]$. Then since H is $(P_2 \cup P_3)$ -free, $H_i \cong K_1$, for each $i \in \{2, 3, \dots, k\}$. So $\alpha(H) = \alpha(G_t) + |V_1| + (k - 1)$. Now, $\theta(H) \leq \theta([V_1]) + \theta([V_2 \cup V_3]) + (k - 1) = |V_1| + \theta([V_2 \cup V_3]) + (k - 1) = \theta([V_2 \cup V_3]) + \alpha(H) - \alpha(G_t)$. It is easily verified that $\theta([V_2 \cup V_3]) \leq \alpha(G_t) + 1$. Hence, $\theta(H) \leq \alpha(H) + 1$, and the corollary is proved. \square

The graphs $C_5(\overline{K}_{n_1}, \overline{K}_{n_2}, \overline{K}_{n_3}, \overline{K}_{n_4}, \overline{K}_{n_5})$ show that the bound in Corollary 3 is tight.

3.4 The class of $(2K_2, H)$ -free graphs, $H \in \{HVN, K_5 - e\}$

In order to prove our next results, we need the following notation. Let G be a connected graph that contains an induced diamond, say D , with vertex set $L_0 := \{v_1, v_2, v_3, v_4\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_2v_4\}$. Define sets $L_1 := \{y \in V(G) \setminus L_0 \mid y \text{ has a neighbor in } L_0\}$ and $L_2 := V(G) \setminus (L_0 \cup L_1)$. Moreover, let:

$$\begin{aligned} X_i &= \{x \in L_1 \mid N(x) \cap L_0 = \{v_i\}\}; i \in [3], \\ Y_1 &= \{x \in L_1 \mid N(x) \cap L_0 = \{v_1, v_2\}\}, \\ Y_2 &= \{x \in L_1 \mid N(x) \cap L_0 = \{v_2, v_3\}\}, \\ Z_1 &= \{x \in L_1 \mid N(x) \cap L_0 = \{v_1, v_3\}\}, \\ Z_2 &= \{x \in L_1 \mid N(x) \cap L_0 = \{v_1, v_2, v_3\}\}. \end{aligned}$$

Then we have the following lemma, and we leave its proof as it can be routinely verified.

Lemma 1 *Let G be a connected $2K_2$ -free graph that contains an induced diamond D . Let L_0 , subsets of L_1 , and L_2 be defined as above. Then the following hold:*

$$(1) \ V(G) = N(v_4) \cup \{v_4\} \cup X_1 \cup X_2 \cup X_3 \cup Y_1 \cup Y_2 \cup Z_1 \cup Z_2 \cup L_2.$$

$$(2) \ \text{We have either } X_1 = \emptyset \text{ or } X_3 = \emptyset.$$

$$(3) \ X_1 \cup X_2 \cup Y_1, Y_2, Z_1 \text{ and } L_2 \text{ are independent sets.}$$

$$(4) \ [X_1 \cup X_2 \cup X_3 \cup Y_1 \cup Y_2 \cup Z_1, L_2] = \emptyset. \quad \square$$

Theorem 4 *Let G be a connected $(2K_2, HVN)$ -free graph. Then G is either a $(2K_2, \text{diamond})$ -free graph or there exists a partition (V_1, \dots, V_4) of $V(G)$ such that*

- (i) V_1 induces a $(2K_2, \text{paw})$ -free graph of G with $\omega([V_1]) \leq \omega(G) - 1$, and
- (ii) V_i is an independent set, for each $i \in \{2, 3, 4\}$.

Proof. Let G be a connected $(2K_2, HVN)$ -free graph. If G is diamond-free, then the theorem holds. Suppose that G contains an induced diamond, say D . We use Lemma 1. By (2) and by symmetry, we may assume that $X_3 = \emptyset$. Now, since G is HVN -free, we have the following:

- For any $v \in V(G)$, $N(v)$ induces a paw-free graph with $\omega([N(v)]) \leq \omega(G) - 1$.
- $Y_2 \cup Z_2$ is an independent set (by using (3)).

Define $V_1 := N(v_4)$, $V_2 := X_1 \cup X_2 \cup Y_1$, $V_3 := Y_2 \cup Z_2$, and $V_4 := Z_1 \cup L_2 \cup \{v_4\}$. Then by (3) and (4), and by the above properties, we see that (V_1, \dots, V_4) is a required partition of $V(G)$, and the theorem is proved. \square

Corollary 4 *Let G be a $(2K_2, HVN)$ -free graph. Then $\chi(G) \leq \omega(G) + 3$.*

Proof. Let G be a connected $(2K_2, HVN)$ -free graph. We use Theorem 4.

If G is a $(2K_2, \text{diamond})$ -free graph, then by (R8), $\chi(G) \leq \omega(G) + 1$, and the corollary holds. Suppose that $V(G)$ admits a partition as in Theorem 4. So, $\chi(G) \leq \chi([V_1]) + 3$. Since $\chi([V_1]) \leq \omega([V_1]) + 1$ (by (R7)), we have $\chi(G) \leq \omega([V_1]) + 1 + 3 \leq \omega(G) + 3$, as desired. \square

Theorem 5 *Let G be a connected $(2K_2, K_5 - e)$ -free graph. Then G is either a $(2K_2, \text{diamond})$ -free graph or there exists a partition (V_1, \dots, V_5) of $V(G)$ such that*

- (i) V_1 induces a $(2K_2, \text{diamond})$ -free graph of G with $\omega([V_1]) \leq \omega(G) - 1$, and
- (ii) V_i is an independent set, for each $i \in \{2, 3, 4, 5\}$.

Proof. Let G be a connected $(2K_2, K_5 - e)$ -free graph. If G is diamond-free, then the theorem holds. Suppose that G contains an induced diamond, say D . We use Lemma 1. By (2) and by symmetry, we may assume that $X_3 = \emptyset$. Now, since G is $(K_5 - e)$ -free, we have the following:

- For any $v \in V(G)$, $N(v)$ induces a diamond-free graph with $\omega([N(v)]) \leq \omega(G) - 1$.
- Z_2 is an independent set.

Define $V_1 := N(v_4)$, $V_2 := X_1 \cup X_2 \cup Y_1$, $V_3 := Y_2$, $V_4 := Z_1 \cup L_2 \cup \{v_4\}$, and $V_5 := Z_2$. Then by (3) and (4), and by the above properties, we see that (V_1, \dots, V_5) is a required partition of $V(G)$, and the theorem is proved. \square

Corollary 5 *Let G be a $(2K_2, K_5 - e)$ -free graph. Then $\chi(G) \leq \omega(G) + 4$.*

Proof. Let G be a connected $(2K_2, K_5 - e)$ -free graph. We use Theorem 5.

If G is a $(2K_2, \text{diamond})$ -free graph, then by (R8), $\chi(G) \leq \omega(G) + 1$, and the corollary holds. Suppose that $V(G)$ admits a partition as in Theorem 5. So, $\chi(G) \leq \chi([V_1]) + 4$. Since $\chi([V_1]) \leq \omega([V_1]) + 1$ (by (R8)), we have $\chi(G) \leq \omega([V_1]) + 1 + 4 \leq \omega(G) + 4$, as desired. \square

3.5 The class of $(2K_2, K_1 + H)$ -free graphs, for any graph H

Theorem 6 *Let H be any graph. Suppose that for every $(2K_2, H)$ -free graph G' , $\chi(G') \leq f(\omega(G'))$. Then for every $(2K_2, K_1 + H)$ -free graph G , we have $\chi(G) \leq 2f(\omega(G) - 1) + 1$.*

Proof. Let G be a $(2K_2, K_1 + H)$ -free graph. If G is an edgeless graph, then the theorem is obvious. So we may assume that there exist adjacent vertices, say v_1 and v_2 in $V(G)$. For each $i \in \{1, 2\}$, let $A_i := \{x \in V(G) \setminus \{v_1, v_2\} \mid N(x) \cap \{v_1, v_2\} = \{v_i\}\}$. Also, let $B := \{x \in V(G) \setminus \{v_1, v_2\} \mid N(x) \cap \{v_1, v_2\} = \{v_1, v_2\}\}$ and $C := V(G) \setminus (\{v_1, v_2\} \cup A_1 \cup A_2 \cup B)$. Then, we have the following:

- (i) Since G does not induce a $K_1 + H$, we have: for any $v \in V(G)$, $N(v)$ induces a H -free graph. So, for each $i \in \{1, 2\}$, $[A_i \cup B]$ is a H -free graph with $\omega([A_i \cup B]) \leq \omega(G) - 1$.
- (ii) Since G does not induce a $2K_2$, we see that C is an independent set.

Now, $\chi(G) \leq \chi([N(v_1)]) + \chi([A_2]) + \chi([C \cup \{v_1\}]) = \chi([A_1 \cup B \cup \{v_2\}]) + \chi([A_2]) + \chi([C \cup \{v_1\}])$. Since for every $(2K_2, H)$ -free graph G' , $\chi(G') \leq f(\omega(G'))$, and since $C \cup \{v_2\}$ is an independent set (by (ii)), we have, by (R4), $\chi(G) \leq f(\omega(G) - 1) + f(\omega(G) - 1) + 1 = 2f(\omega(G) - 1) + 1$ (by (i)), as desired. \square

Corollary 6 *Let G be a $(2K_2, K_5)$ -free graph. Then $\chi(G) \leq 2\omega(G) + 1 \leq 9$.*

Proof. Since G is a $(2K_2, K_1 + K_4)$ -free graph, and since for every $(2K_2, K_4)$ -free graph G' , $\chi(G') \leq \omega(G') + 1 \leq 4$ (see [10]), the corollary follows by Theorem 6. \square

4 Superclasses of $2K_2$ -free graphs

In this section, we show that some superclasses of $2K_2$ -free graphs are χ -bounded.

If G is a graph and if $e := uv$ is an edge in G , then we simply write $A(e)$ to denote the set of all vertices in G that are not adjacent to both u and v in G . The proof of the following theorem is very similar to the proof of Wagon [19] for the class of $2K_2$ -free graphs, and we give it here for completeness.

Theorem 7 *Let \mathcal{H} be a class of graphs and let G be any graph. Suppose that \mathcal{H} is χ -bounded with χ -binding function f . Suppose that for every edge e in G , $[A(e)] \in \mathcal{H}$. Then $\chi(G) \leq \binom{\omega(G)}{2} \cdot f(\omega(G)) + \omega(G)$.*

Proof. Let $\omega := \omega(G)$ and let K be a complete subgraph of G with $|K| = \omega$, and $V(K) = \{v_1, v_2, \dots, v_\omega\}$. Then every vertex in $x \in V(G) \setminus V(K)$ is not adjacent to at least one vertex in K . Otherwise, $\{x\} \cup V(K)$ induces a clique of size larger than ω which is a contradiction. For each $i, j \in [\omega]$, $i \neq j$, let $A_{ij} := A(e_{ij})$, where e_{ij} is the edge $v_i v_j$, and let $B_i := \{x \in V(G) \setminus V(K) \mid [\{x\}, V(K) \setminus \{v_i\}] \text{ is complete} \}$. Moreover, let $A := \cup A_{ij}$ and $B := \cup B_i$. Then $V(G) = V(K) \cup A \cup B$.

Now, for each $i, j \in [\omega]$, $i \neq j$, we have:

- (i) Since for every edge e in G , $[A(e)] \in \mathcal{H}$, we have $[A_{ij}] \in \mathcal{H}$.
- (ii) $B_i \cup \{v_i\}$ is an independent set. If not, then there exist adjacent vertices, say x and y in B_i . But, then $\{x, y\} \cup (V(K) \setminus \{v_i\})$ induces a clique of size $\omega + 1$, a contradiction.

So, $\chi(G) \leq \sum_{\{i,j\} \subseteq [\omega]} \chi([A(e_{ij})]) + \sum_{i=1}^{\omega} \chi([B_i \cup \{v_i\}])$. Then by (i) and (ii), and by (R4), $\chi(G) \leq \sum_{\{i,j\} \subseteq [\omega]} f(\omega([A(e_{ij})])) + \sum_{i=1}^{\omega} \chi([B_i \cup \{v_i\}])$. Then since $\omega([A(e_{ij})]) \leq \omega$ and since $B_i \cup \{v_i\}$ is an independent set, for each $i, j \in [\omega]$, $i \neq j$, we have $\chi(G) \leq \binom{\omega}{2} \cdot f(\omega) + \omega$, and the theorem is proved. \square

Then we immediately have the following.

Corollary 7 ([19]) *Let G be a $2K_2$ -free graph. Then $\chi(G) \leq \binom{\omega(G)+1}{2}$.*

Proof. Since G is $2K_2$ -free, for each $i, j \in [\omega]$, $i \neq j$, $A(e_{ij})$ is an independent set in G . So, $\omega([A(e_{ij})]) \leq 1$, and hence the corollary follows from the proof of Theorem 7. \square

Corollary 8 *Let G be any graph. If for every edge e in G , $A(e)$ induces a perfect graph, then $\chi(G) \leq \frac{\omega(G)^3 - \omega(G)^2 + 2\omega(G)}{2}$.* \square

Corollary 9 *Let G be a $(P_2 \cup P_4)$ -free graph. Then $\chi(G) \leq \frac{\omega(G)^3 - \omega(G)^2 + 2\omega(G)}{2}$.*

Proof. Since every P_4 -free is perfect (by (R2)), the corollary follows from Corollary 8. \square

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